

NORMALITY AND NON-NORMALITY OF GROUP COMPACTIFICATIONS IN SIMPLE PROJECTIVE SPACES

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ABSTRACT. If G is a complex simply connected semisimple algebraic group and if λ is a dominant weight, we consider the compactification $X_\lambda \subset \mathbb{P}(\text{End}(V(\lambda)))$ obtained as the closure of the $G \times G$ -orbit of the identity and we give necessary and sufficient conditions on the support of λ so that X_λ is normal; as well, we give necessary and sufficient conditions on the support of λ so that X_λ is smooth.

INTRODUCTION

Consider a semisimple simply connected algebraic group G over an algebraically closed field \mathbb{k} of characteristic zero. If λ is a dominant weight (with respect to a fixed maximal torus T and a fixed Borel subgroup $B \supset T$) and if $V(\lambda)$ is the simple G -module of highest weight λ , then $\text{End}(V(\lambda))$ is a simple $G \times G$ -module. Let $I_\lambda \in \text{End}(V(\lambda))$ be the identity map and consider the variety $X_\lambda \subset \mathbb{P}(\text{End}(V(\lambda)))$ given by the closure of the $G \times G$ -orbit of $[I_\lambda]$. In [Ka], S. Kannan studied for which λ this variety is projectively normal, and this happens precisely when λ is minuscule. In [Ti], D. Timashev studied the more general situation of a sum of irreducible representations, giving necessary and sufficient conditions for the normality and smoothness of these compactifications; however the conditions for normality are not completely explicit. In this paper we give an explicit characterization of the normality of X_λ , which allows to simplify the conditions for the smoothness as well.

To explain our results we need some notation. Let Δ be the set of simple roots (w.r.t. $T \subset B$) and identify Δ with the vertices of the Dynkin diagram. Define the support of λ as the set $\text{Supp}(\lambda) = \{\alpha \in \Delta : \langle \lambda, \alpha^\vee \rangle \neq 0\}$.

Theorem A (see Theorem 13). *The variety X_λ is normal if and only if λ satisfies the following property:*

- (\star) *For every non-simply laced connected component Δ' of Δ , if $\text{Supp}(\lambda) \cap \Delta'$ contains a long root, then it contains also the short root which is adjacent to a long simple root.*

In particular, if the Dynkin diagram of G is simply laced then X_λ is normal, for all λ . In the paper we will prove the theorem in a more general form, for simple (i.e. with a unique closed orbit) linear projective compactifications of an adjoint group (see section 1.4). We will make use of the wonderful compactification of G_{ad} , the adjoint group of G , and of the results on projective normality of these compactifications proved by S. Kannan in [Ka]. These results hold in the more general case of a symmetric variety; however our method does not apply to this more general situation (see section 4.2).

Theorem B (see Theorem 22). *The variety X_λ is smooth if and only if λ satisfies property (\star) of Theorem A together with the following properties:*

- i) *For every connected component Δ' of Δ , $\text{Supp}(\lambda) \cap \Delta'$ is connected and, in case it contains a unique element, then this element is an extreme of Δ' ;*
- ii) *$\text{Supp}(\lambda)$ contains every simple root which is adjacent to three other simple roots and at least two of the latter;*
- iii) *Every connected component of $\Delta \setminus \text{Supp}(\lambda)$ is of type A.*

Theorem B can be generalized to any simple and normal adjoint symmetric variety. Following a criterion of \mathbb{Q} -factoriality for spherical varieties given by M. Brion in [Br], properties i) and ii) characterize

the \mathbb{Q} -factoriality of the normalization of X_λ (see Proposition 20), while property iii) arises from a criterion of smoothness given by D. Timashev in [Ti] in the case of a linear projective compactification of a reductive group.

As a corollary of Theorem B, we get that X_λ is smooth if and only if its normalization is smooth.

The paper is organized as follows. In the first section we introduce the wonderful compactification of G_{ad} and the normalization of the variety X_λ . In the second section we prove Theorem A, and in the third section Theorem B. In the last section we discuss some possible generalizations of our results.

1. PRELIMINARIES

1.1. Notation. Recall that G is semisimple and simply connected. Fix a Borel subgroup $B \subset G$, a maximal torus $T \subset B$ and let U denote the unipotent radical of B . Lie algebras of groups denoted by upper-case latin letters (G, U, L, \dots) will be denoted by the corresponding lower-case german letter ($\mathfrak{g}, \mathfrak{u}, \mathfrak{l}, \dots$). Let Φ denote the set of roots of G relatively to T and $\Delta \subset \Phi$ the basis associated to the choice of B . For all $\alpha \in \Delta$ let $e_\alpha, \alpha^\vee, f_\alpha$ be an $\mathfrak{sl}(2)$ -triple of T -weights $\alpha, 0, -\alpha$. Let Λ denote the weight lattice of T and Λ^+ the subset of dominant weights. For all $\alpha \in \Delta$, denote by ω_α the corresponding fundamental weight.

If $\lambda \in \Lambda$, recall the definition of its *support*:

$$\text{Supp}(\lambda) = \{\alpha \in \Delta : \langle \lambda, \alpha^\vee \rangle \neq 0\}.$$

If $I \subset \Delta$, define its *border* ∂I , its *interior* I° and its *closure* \bar{I} as follows:

$$\partial I = \{\alpha \in \Delta \setminus I : \exists \beta \in I \text{ such that } \langle \beta, \alpha^\vee \rangle \neq 0\};$$

$$I^\circ = I \setminus \partial(\Delta \setminus I);$$

$$\bar{I} = I \cup \partial I.$$

For $\lambda \in \Lambda$, denote by \mathcal{L}_λ the line bundle on G/B whose T -weight in the point fixed by B is $-\lambda$. For λ dominant, $V(\lambda) = \Gamma(G/B, \mathcal{L}_\lambda)^*$ is an irreducible G -module of highest weight λ ; when we deal with different groups we will use the notation $V_G(\lambda)$.

Denote by $\Pi(\lambda)$ the set of weights occurring in $V(\lambda)$ and set $\Pi^+(\lambda) = \Pi(\lambda) \cap \Lambda^+$. Let $\lambda \mapsto \lambda^*$ be the linear involution of Λ defined by $(V(\lambda))^* \simeq V(\lambda^*)$, for any dominant weight λ .

The weight lattice Λ is endowed with the dominance order \leq defined as follows: $\mu \leq \lambda$ if and only if $\lambda - \mu \in \mathbb{N}\Delta$. If $\beta = \sum_{\alpha \in \Delta} n_\alpha \alpha \in \mathbb{Z}\Delta$, define its *support over Δ* (not to be confused with the previous one) as follows:

$$\text{Supp}_\Delta(\beta) = \{\alpha \in \Delta : n_\alpha \neq 0\}.$$

We introduce also some notations about the multiplication of sections. Notice that, for all $\lambda, \mu \in \Lambda$, $\mathcal{L}_\lambda \otimes \mathcal{L}_\mu = \mathcal{L}_{\lambda+\mu}$. Therefore, if λ, μ are dominant weights and $n \in \mathbb{N}$, the multiplication of sections defines maps as follows:

$$m_{\lambda, \mu} : V(\lambda) \times V(\mu) \rightarrow V(\lambda + \mu) \quad \text{and} \quad m_\lambda^n : V(\lambda) \rightarrow V(n\lambda).$$

We will also write uv for $m_{\lambda, \mu}(u, v)$ and u^n for $m_\lambda^n(u)$. Since G/B is irreducible, $m_{\lambda, \mu}$ and m_λ^n induce the following maps at the level of projective spaces:

$$\psi_{\lambda, \mu} : \mathbb{P}(V(\lambda)) \times \mathbb{P}(V(\mu)) \rightarrow \mathbb{P}(V(\lambda + \mu)) \quad \text{and} \quad \psi_\lambda^n : \mathbb{P}(V(\lambda)) \rightarrow \mathbb{P}(V(n\lambda)).$$

The following lemma is certainly well known; however we do not know any reference.

Lemma 1. *Let λ, μ be dominant weights.*

- i) *If $\text{Supp}(\lambda) \cap \text{Supp}(\mu) = \emptyset$, then the map $\psi_{\lambda, \mu} : \mathbb{P}(V(\lambda)) \times \mathbb{P}(V(\mu)) \rightarrow \mathbb{P}(V(\lambda + \mu))$ is a closed embedding.*
- ii) *For any $n > 0$, the map $\psi_\lambda^n : \mathbb{P}(V(\lambda)) \rightarrow \mathbb{P}(V(n\lambda))$ is a closed embedding.*

Proof. i). Fix highest weight vectors $v_\lambda \in V(\lambda)$, $v_\mu \in V(\mu)$ and $v_{\lambda+\mu} = v_\lambda v_\mu \in V(\lambda + \mu)$.

If V is irreducible, then $\mathbb{P}(V)$ has a unique closed orbit, namely the orbit of the highest weight vector. Consequently, since $\mathbb{P}(V(\lambda)) \times \mathbb{P}(V(\mu))$ has a unique closed orbit, in order to prove the claim it suffices to prove that $\psi_{\lambda,\mu}$ is smooth in $x = ([v_\lambda], [v_\mu])$ and that the inverse image of $[v_{\lambda+\mu}]$ is x . The second claim is clear for weight reasons.

In order to prove that $\psi_{\lambda,\mu}$ is smooth in x , consider T -stable complements $U \subset V(\lambda)$, $V \subset V(\mu)$ and $W \subset V(\lambda + \mu)$ of $\mathbb{k}v_\lambda$, $\mathbb{k}v_\mu$ and $\mathbb{k}v_{\lambda+\mu}$. So in a neighbourhood of x the map $\psi_{\lambda,\mu}$ can be described as

$$\psi : U \times V \longrightarrow W \quad \text{where } \psi(u, v) = uv_\mu + v_\lambda v + uv.$$

The differential of $\psi_{\lambda,\mu}$ in x is then given by the differential of ψ in $(0, 0)$, thus it is described as follows:

$$d\psi_x(u, v) = uv_\mu + v_\lambda v.$$

Suppose that $d\psi_x$ is not injective. Since it is T -equivariant, consider a maximal weight $\eta \in \Pi(\lambda + \mu) \setminus \{\lambda + \mu\}$ such that there exists a couple of non-zero T -eigenvectors $(u, v) \in \ker d\psi_x$ with weights respectively $\eta - \mu$ and $\eta - \lambda$. Suppose that $\eta - \mu \in \Pi(\lambda) \setminus \{\lambda\}$ is not maximal and take $\alpha \in \Delta$ such that $\eta - \mu + \alpha \in \Pi(\lambda) \setminus \{\lambda\}$ and $e_\alpha u \neq 0$: then

$$(e_\alpha u)v_\mu + v_\lambda(e_\alpha v) = e_\alpha(uv_\mu + v_\lambda v) = 0$$

and $\eta + \alpha \in \Pi(\lambda + \mu) \setminus \{\lambda + \mu\}$, against the maximality of η . Thus $\eta - \mu$ is maximal in $\Pi(\lambda) \setminus \{\lambda\}$ and similarly $\eta - \lambda$ is maximal in $\Pi(\mu) \setminus \{\mu\}$. Therefore, on one hand it must be

$$\eta - \mu = \lambda - \alpha$$

with $\alpha \in \text{Supp}(\lambda)$, while on the other hand it must be

$$\eta - \lambda = \mu - \beta$$

with $\beta \in \text{Supp}(\mu)$. Since $\text{Supp}(\lambda) \cap \text{Supp}(\mu) = \emptyset$, this is impossible and shows that, if $(u, v) \in \ker d\psi_x$, then it must be $u = 0$ or $v = 0$. Suppose now that $(u, 0) \in \ker d\psi_x$: then $uv_\mu = 0$ and by the irreducibility of G/B also $u = 0$. A similar argument applies if $v = 0$.

ii). Suppose that $v, w \in V(\lambda)$ are such that $v^n = w^n$: then $v = tw$ for some $t \in \mathbb{k}$. Thus ψ_λ^n is injective. Let us show now that ψ_λ^n is smooth; it is enough to show it in $x = [v_\lambda]$ where $v_\lambda \in V(\lambda)$ is a highest weight vector. Let $V \subset V(\lambda)$ be the T -stable complement of $\mathbb{k}v_\lambda$, identified with the tangent space $T_x\mathbb{P}(V(\lambda))$. If $v \in V$, the differential $d(\psi_\lambda^n)_x$ is described as follows

$$d(\psi_\lambda^n)_x(v) = nv_\lambda^{n-1}v.$$

Thus $d(\psi_\lambda^n)_x$ is injective and ψ_λ^n is smooth. □

1.2. The variety X_λ . If λ is a dominant weight, denote by $E(\lambda)$ the $G \times G$ -module $\text{End}(V(\lambda))$ and set X_λ the closure of the $G \times G$ -orbit of $[I_\lambda] \in \mathbb{P}(E(\lambda))$. More generally if $\lambda_1, \dots, \lambda_m$ are dominant weights we define

$$X_{\lambda_1, \dots, \lambda_m} = \overline{G \times G([I_{\lambda_1}], \dots, [I_{\lambda_m}])} \subset \mathbb{P}(E(\lambda_1)) \times \dots \times \mathbb{P}(E(\lambda_m)).$$

Since $E(\lambda)$ is an irreducible $G \times G$ -module of highest weight (λ, λ^*) , as a consequence of Lemma 1 we get that if λ and μ have non-intersecting supports and if $n \in \mathbb{N}$ then

$$X_{\lambda+\mu} \simeq X_{\lambda, \mu} \quad \text{and} \quad X_{n\lambda} \simeq X_\lambda.$$

As a consequence we get the following proposition:

Proposition 2. *Let λ, μ be dominant weights. Then $X_\lambda \simeq X_\mu$ as $G \times G$ -varieties if and only if λ and μ have the same support. Moreover, if $\text{Supp}(\lambda) = \{\alpha_1, \dots, \alpha_m\}$ then*

$$X_\lambda \simeq X_{\omega_{\alpha_1}, \dots, \omega_{\alpha_m}}.$$

Proof. By the discussion above we have to prove only that the condition is necessary. This follows by noticing that if X_λ and X_μ are $G \times G$ -isomorphic then also their closed $G \times G$ -orbits are isomorphic, which is equivalent to the fact that λ and μ have the same support. \square

1.3. The wonderful compactification of G_{ad} and the normalization of X_λ . When λ is a regular weight (i.e. $\text{Supp}(\lambda) = \Delta$) the variety X_λ is called the wonderful compactification of G_{ad} and it has been studied by C. De Concini and C. Procesi in [DP]. We will denote this variety by M : it is smooth and the complement of its open orbit is the union of smooth prime divisors with normal crossings whose intersection is the closed orbit. The closed orbit of M is isomorphic to $G/B \times G/B$ and the restriction of line bundles determines an embedding of $\text{Pic}(M)$ into $\text{Pic}(G/B \times G/B)$, that we identify with $\Lambda \times \Lambda$ as before; the image of this map is the set of weights of the form (λ, λ^*) . Therefore $\text{Pic}(M)$ is identified with Λ and we denote by \mathcal{M}_λ a line bundle on M whose restriction to $G/B \times G/B$ is isomorphic to $\mathcal{L}_\lambda \boxtimes \mathcal{L}_{\lambda^*}$. If $D \subset M$ is a $G \times G$ -stable prime divisor then the line bundle defined by D is of the form \mathcal{M}_{α_D} , where α_D is a simple root. The map $D \mapsto \alpha_D$ defines a bijection between the set of $G \times G$ -stable prime divisors and Δ , and we denote by M_α the prime divisor which corresponds to a simple root α . We denote by s_α a section of \mathcal{M}_α whose associated divisor is M_α ; notice that such a section is $G \times G$ -invariant. More generally if $\nu = \sum_{\alpha \in \Delta} n_\alpha \alpha \in \mathbb{N}\Delta$, set $s^\nu = \prod_{\alpha \in \Delta} s_\alpha^{n_\alpha} \in \Gamma(M, \mathcal{M}_\nu)$. Then, given any $\lambda \in \Lambda$, the multiplication by s^ν injects $\Gamma(M, \mathcal{M}_{\lambda-\nu})$ in $\Gamma(M, \mathcal{M}_\lambda)$.

If λ is a dominant weight, the map $G_{\text{ad}} \rightarrow \mathbb{P}(E(\lambda))$ extends to a map $q_\lambda : M \rightarrow \mathbb{P}(E(\lambda))$ (see [DP]) whose image is X_λ and such that $\mathcal{M}_\lambda = q_\lambda^*(\mathcal{O}_{\mathbb{P}(E(\lambda))}(1))$. If we pull back the homogeneous coordinates of $\mathbb{P}(E(\lambda))$ to M , we get then a submodule of $\Gamma(M, \mathcal{M}_\lambda)$ which is isomorphic to $E(\lambda)^*$; by abuse of notation we will denote this submodule by $E(\lambda)^*$.

If $\lambda \in \Lambda$, in [DP, Theorem 8.3] the following decomposition of $\Gamma(M, \mathcal{M}_\lambda)$ is given:

$$\Gamma(M, \mathcal{M}_\lambda) = \bigoplus_{\mu \in \Lambda^+ : \mu \leq \lambda} s^{\lambda-\mu} E(\mu)^*.$$

Consider the graded algebra $A(\lambda) = \bigoplus_{n=0}^{\infty} A_n(\lambda)$, where $A_n(\lambda) = \Gamma(M, \mathcal{M}_{n\lambda})$, and set $\tilde{X}_\lambda = \text{Proj } A(\lambda)$. We have then a commutative diagram as follows:

$$\begin{array}{ccc} M & \xrightarrow{p_\lambda} & \tilde{X}_\lambda \\ & \searrow q_\lambda & \downarrow r_\lambda \\ & & X_\lambda \end{array}$$

In [Ka], it has been shown that $A(\lambda)$ is generated in degree 1 and in [D] that $r = r_\lambda$ is the normalization of X_λ . Notice that the projective coordinate ring of $X_\lambda \subset \mathbb{P}(E(\lambda))$ is given by the graded subalgebra $B(\lambda) = \bigoplus_{n=0}^{\infty} B_n(\lambda)$ of $A(\lambda)$ generated by $E(\lambda)^* \subset \Gamma(M, \mathcal{M}_\lambda)$.

1.4. The variety X_Σ . We consider now a generalization of the variety X_λ . Let Σ be a finite set of dominant weights and denote $E(\Sigma) = \bigoplus_{\mu \in \Sigma} E(\mu)$; let $x_\Sigma = [(I_\mu)_{\mu \in \Sigma}] \in \mathbb{P}(E(\Sigma))$ and define X_Σ as the closure of the $G \times G$ -orbit of x_Σ in $\mathbb{P}(E(\Sigma))$. If $\Sigma = \{\lambda\}$, then we get the variety X_λ , while if $\Sigma = \Pi^+(\lambda)$ we get its normalization \tilde{X}_λ . Notice that the diagonal action of G fixes the point x_Σ so we have a $G \times G$ equivariant map $G \rightarrow X_\Sigma$ given by $g \mapsto (g, 1)x_\Sigma$. This map induces a map from G_{ad} to X_Σ if and only if the action of the center of $G \times G$ on $E(\lambda)$ is the same for all $\lambda \in \Sigma$ or equivalently if Σ is contained in a coset of Λ modulo $\mathbb{Z}\Delta$. In this case we say that X_Σ is a *semi-compactification* of G_{ad} . If G_{ad} is a simple group and $\Sigma \neq \{0\}$ then X_Σ is a compactification of G_{ad} , while if G_{ad} is not simple we can only say that is a compactification of a group which is a quotient of G_{ad} .

We say that Σ is *simple* if there exists $\lambda \in \Sigma$ such that $\Sigma \subset \Pi^+(\lambda)$ or equivalently if Σ contains a unique maximal element with respect to the dominance order \leq . Notice also that if $\lambda \in \Sigma$ is such that for all $\mu \in \Sigma$ different from λ the vector $\mu - \lambda$ is not in $\mathbb{Q}_{\geq 0}[\Delta]$ then is easy to construct a cocharacter

$\chi : \mathbb{k}^* \longrightarrow G \times G$ such that $\lim_{t \rightarrow 0} \chi(t)x_\Sigma$ is the highest weight line in $\mathbb{P}(E(\lambda))$. In particular X_Σ is a simple $G \times G$ semi-compactification of G_{ad} if and only if Σ is simple.

By the description of the normalization of X_λ is Σ is simple and $\lambda \in \Sigma$ is the maximal element, then we get

$$\tilde{X}_\lambda \xrightarrow{r} X_\Sigma \longrightarrow X_\lambda$$

In particular, it follows that $r = r_\Sigma : \tilde{X}_\lambda \longrightarrow X_\Sigma$ is the normalization of X_Σ .

If Σ is simple, denote $B(\Sigma) = \bigoplus_{n=0}^\infty B_n(\Sigma)$ the projective coordinate ring of $X_\Sigma \subset \mathbb{P}(E(\Sigma))$: it is the subalgebra of $A(\lambda)$ generated by $E(\Sigma)^* \subset \Gamma(M, \mathcal{M}_\lambda)$.

Remark 3. The discussion above and the fact that in $\mathbb{P}(E(\lambda))$ there is only one point fixed by the diagonal action of G (the line of scalar matrices) proves that any $G \times G$ linear projective compactification of G_{ad} is of the form X_Σ . A projective $G \times G$ -variety X is said to be *linear* if there exists an equivariant embedding $X \subset \mathbb{P}(V)$ where V is a finite dimensional rational $G \times G$ -module. In particular as a consequence of Sumihiro's Theorem (see for example [KKLV, Corollary 2.6]) all normal projective compactifications are linear. In this paper we study only linear compactifications.

2. NORMALITY

In this section we determine for which simple Σ the variety X_Σ is normal, proving in particular Theorem A. In the following, by λ we will always denote the maximal element of Σ .

Let $\varphi_\lambda \in E(\lambda)^*$ be a highest weight vector and set $X_\Sigma^\circ \subset X_\Sigma$ the open affine subset defined by the non-vanishing of φ_λ . In particular, we set $\tilde{X}_\lambda = X_{\Pi^+(\lambda)}$ and notice that $\tilde{X}_\lambda^\circ = r^{-1}(X_\Sigma^\circ)$. Notice that X_Σ° is $B \times B$ -stable and, since it intersects the closed orbit, it intersects every orbit: therefore X_Σ is normal if and only if X_Σ° is normal if and only if the restriction $r|_{\tilde{X}_\lambda^\circ} : \tilde{X}_\lambda^\circ \rightarrow X_\Sigma^\circ$ is an isomorphism. Denote by $\bar{B}(\Sigma)$ the coordinate ring of X_Σ° and by $\bar{A}(\lambda)$ the coordinate ring of \tilde{X}_λ° ; then we have

$$\bar{A}(\lambda) = \left\{ \frac{\varphi}{\varphi_\lambda^n} : \varphi \in A_n(\lambda) \right\} \supset \left\{ \frac{\varphi}{\varphi_\lambda^n} : \varphi \in B_n(\Sigma) \right\} = \bar{B}(\Sigma)$$

and X_Σ is normal if and only if $\bar{A}(\lambda) = \bar{B}(\Sigma)$. The rings $\bar{A}(\lambda)$ and $\bar{B}(\Sigma)$ are not $G \times G$ -modules, however since X_Σ° is an open subset of X_Σ we still have an action of the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ on them.

By [Ka], $\bar{A}(\lambda)$ is generated by the elements of the form φ/φ_λ with $\varphi \in A_1(\lambda)$. In particular we have the following lemma.

Lemma 4. *The variety X_Σ is normal if and only if for all $\mu \in \Lambda^+$ such that $\mu \leq \lambda$ there exists $n > 0$ such that*

$$s^{\lambda-\mu} E(\mu + (n-1)\lambda)^* \subset B_n(\Sigma).$$

Proof. Let $\varphi_\mu \in s^{\lambda-\mu} E(\mu)^*$ be a highest weight vector and suppose that X_Σ is normal. Then, by the descriptions of $\bar{A}(\lambda)$ and $\bar{B}(\Sigma)$, for every dominant weight $\mu \leq \lambda$ there exist $n > 0$ and $\varphi \in B_n(\Sigma)$ such that $\varphi/\varphi_\lambda^n = \varphi_\mu/\varphi_\lambda$ or equivalently $\varphi = \varphi_\mu \varphi_\lambda^{n-1} \in B_n(\Sigma)$. Since φ is a highest weight vector of the module $s^{\lambda-\mu} E(\mu + (n-1)\lambda)^*$ the claim follows.

Conversely assume that for every dominant weight $\mu \leq \lambda$ there exists n such that

$$s^{\lambda-\mu} E(\mu + (n-1)\lambda)^* \subset B_n(\Sigma);$$

in particular $\varphi = \varphi_\mu \varphi_\lambda^{n-1} \in B_n(\Sigma)$. Let's prove that $\varphi/\varphi_\lambda \in \bar{B}(\Sigma)$ for every $\varphi \in s^{\lambda-\mu} E(\mu)^*$; this implies the thesis since $\bar{A}(\lambda)$ is generated in degree one. If $\varphi = \varphi_\mu$ this is clear. Using the action of the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ on $\bar{B}(\Sigma)$, let's show that if $\varphi/\varphi_\lambda \in \bar{B}(\Sigma)$ then $f_\alpha(\varphi)/\varphi_\lambda \in \bar{B}(\Sigma)$: indeed we have

$$\frac{f_\alpha(\varphi)}{\varphi_\lambda} = f_\alpha\left(\frac{\varphi}{\varphi_\lambda}\right) + \frac{\varphi}{\varphi_\lambda} \cdot \frac{f_\alpha(\varphi_\lambda)}{\varphi_\lambda}$$

and the claim follows since $f_\alpha(\varphi_\lambda) \in E(\lambda)^* \subset B_1(\Sigma)$. □

We can describe the set $B_n(\Sigma)$ more explicitly. Indeed, as in [D] or in [Ka], it is possible to identify sections of a line bundle on M with functions on G and use the description of the multiplication of matrix coefficients. Recall that as a $G \times G$ -module we have $\mathbb{k}[G] = \bigoplus_{\lambda \in \Lambda^+} E(\lambda)^* \simeq \bigoplus_{\lambda \in \Lambda^+} V(\lambda)^* \otimes V(\lambda)$. More explicitly if V is a representation of G , define $c_V : V^* \otimes V \rightarrow \mathbb{k}[G]$ as usual by $c_V(\psi \otimes v)(g) = \langle \psi, gv \rangle$. If we multiply functions in $\mathbb{k}[G]$ of this type then we get

$$c_V(\psi \otimes v) \cdot c_W(\chi \otimes w) = c_{V \otimes W}((\psi \otimes \chi) \otimes (v \otimes w)) :$$

in particular we get that the image of the multiplication $E(\lambda)^* \otimes E(\mu)^* \rightarrow \mathbb{k}[G]$ is the sum of all $E(\nu)^*$ with $V(\nu) \subset V(\lambda) \otimes V(\mu)$.

As a consequence we obtain the following Lemma:

Lemma 5 ([Ka, Lemma 3.1] or [D]). *Let ν, ν' be dominant weights, then the image of $E(\nu)^* \otimes E(\nu')^*$ in $\Gamma(M, \mathcal{M}_{\nu+\nu'})$ via the multiplication map is*

$$\bigoplus_{V(\mu) \subset V(\nu) \otimes V(\nu')} s^{\nu+\nu'-\mu} E(\mu)^*.$$

Proof. Indeed let $\pi : G \rightarrow M$ be the map induced by the inclusion $G_{\text{ad}} \subset M$. Then any line bundle on G can be trivialized so that the image of $\pi^* : E(\lambda)^* \subset \Gamma(M, \mathcal{M}_\nu) \rightarrow \mathbb{k}[G]$ is the image of $c_{V(\lambda)}$ and the claim follows from previous remarks. \square

Together with Lemma 4, this gives the following

Proposition 6. *The variety X_Σ is normal if and only if, for every $\mu \in \Lambda^+$ such that $\mu \leq \lambda$, there exist $n > 0$ and $\lambda_1, \dots, \lambda_n \in \Sigma$ such that*

$$V(\mu + (n-1)\lambda) \subset V(\lambda_1) \otimes \dots \otimes V(\lambda_n).$$

2.1. Remarks on tensor products. By Proposition 6, in order to establish the normality (or the non-normality) of X_Σ , we need some results on tensor product decomposition.

Lemma 7. *Let λ, μ, ν be dominant weights and let $\Delta' \subset \Delta$ be such that $\text{Supp}_\Delta(\lambda + \mu - \nu) \subset \Delta'$; let $L \subset G$ be the standard Levi subgroup associated to Δ' . If $\pi \in \Lambda^+$, denote by $V_L(\pi)$ the simple L -module of highest weight π . Then*

$$V(\nu) \subset V(\lambda) \otimes V(\mu) \iff V_L(\nu) \subset V_L(\lambda) \otimes V_L(\mu).$$

Proof. If \mathfrak{a} is any Lie algebra, denote $\mathfrak{U}(\mathfrak{a})$ the corresponding universal enveloping algebra.

Suppose that $V_L(\nu) \subset V_L(\lambda) \otimes V_L(\mu)$; fix maximal vectors $v_\lambda \in V_L(\lambda)$ and $v_\mu \in V_L(\mu)$ for the Borel subgroup $B \cap L \subset L$ and fix $p \in \mathfrak{U}(\mathfrak{l} \cap \mathfrak{u}^-) \otimes \mathfrak{U}(\mathfrak{l} \cap \mathfrak{u}^-)$ such that $p(v_\lambda \otimes v_\mu) \in V_L(\lambda) \otimes V_L(\mu)$ is a maximal vector of weight ν . Since $V_L(\lambda) \otimes V_L(\mu) \subset V(\lambda) \otimes V(\mu)$, we only need to prove that $p(v_\lambda \otimes v_\mu)$ is a maximal vector for B too. If $\alpha \in \Delta'$ then we have $e_\alpha p(v_\lambda \otimes v_\mu) = 0$ by hypothesis. On the other hand, if $\alpha \in \Delta \setminus \Delta'$, notice that e_α commutes with p , since by its definition p is supported only on the f_α 's with $\alpha \in \Delta'$. Since $v_\lambda \otimes v_\mu$ is a maximal vector for B , then we get

$$e_\alpha p(v_\lambda \otimes v_\mu) = p e_\alpha(v_\lambda \otimes v_\mu) = 0;$$

thus $p(v_\lambda \otimes v_\mu)$ generates a simple G -module of highest weight ν .

Assume conversely that $V(\nu) \subset V(\lambda) \otimes V(\mu)$ and fix $p \in \mathfrak{U}(\mathfrak{u}^-) \otimes \mathfrak{U}(\mathfrak{u}^-)$ such that $p(v_\lambda \otimes v_\mu) \in V(\lambda) \otimes V(\mu)$ is a maximal vector of weight ν . Since $\text{Supp}_\Delta(\lambda + \mu - \nu) \subset \Delta'$, we may assume that the only f_α 's appearing in p are those with $\alpha \in \Delta'$; therefore $p(v_\lambda \otimes v_\mu) \in V_L(\lambda) \otimes V_L(\mu)$ and it generates a simple L -module of highest weight ν . \square

Lemma 8. *Fix $\lambda, \mu, \nu \in \Lambda^+$ such that $V(\nu) \subset V(\lambda) \otimes V(\mu)$. Then, for any $\nu' \in \Lambda^+$, it also holds*

$$V(\nu + \nu') \subset V(\lambda + \nu') \otimes V(\mu).$$

TABLE 1.

type of Φ	highest short root
A_r	$\alpha_1 + \cdots + \alpha_r = \omega_1 + \omega_r$
B_r	$\alpha_1 + \cdots + \alpha_r = \omega_1$
C_r	$\alpha_1 + 2(\alpha_2 + \cdots + \alpha_{r-1}) + \alpha_r = \omega_2$
D_r	$\alpha_1 + 2(\alpha_2 + \cdots + \alpha_{r-2}) + \alpha_{r-1} + \alpha_r = \omega_2$
E_6	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 = \omega_2$
E_7	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 = \omega_1$
E_8	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8 = \omega_8$
F_4	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 = \omega_4$
G_2	$2\alpha_1 + \alpha_2 = \omega_1$

Proof. Fix a maximal vector $v_{\nu'} \in V(\nu')$ and consider the U -equivariant map

$$\begin{aligned} \phi: V(\lambda) \otimes V(\mu) &\longrightarrow V(\lambda + \nu') \otimes V(\mu) \\ w_1 \otimes w_2 &\longmapsto m_{\lambda, \nu'}(w_1, v_{\nu'}) \otimes w_2 \end{aligned}$$

The claim follows since, if $v_{\nu} \in V(\lambda) \otimes V(\mu)$ is a U -invariant vector of weight ν , then $\phi(v_{\nu}) \in V(\lambda + \nu') \otimes V(\mu)$ is a U -invariant vector of weight $\nu + \nu'$. \square

We now describe some more explicit results. When we deal with explicit irreducible root systems, unless otherwise stated, we always use the numbering of simple roots and fundamental weights of Bourbaki [Bo].

In order to describe the simple subsets $\Sigma \subset \Lambda^+$ which give rise to a non-normal variety X_{Σ} , we will make use of following lemma.

Lemma 9.

- (1) Let G be of type B_r . Then, for any n , $V((n-1)\omega_1) \not\subset V(\omega_1)^{\otimes n}$.
- (2) Let G be of type G_2 . Then, for any n , $V(\omega_1 + (n-1)\omega_2) \not\subset V(\omega_2)^{\otimes n}$.

Proof. We consider only the first case, the second is similar. Fix a highest weight vector $v_1 \in V(\omega_1)$. If α is any simple root and if $1 \leq s \leq r$, notice that f_{α} acts non-trivially on $f_{\alpha_{s-1}} \cdots f_{\alpha_1} v_1$ if and only if $\alpha = \alpha_s$. The T -eigenspace of weight 0 in $V(\omega_1)$ is spanned by $v_0 = f_{\alpha_r} \cdots f_{\alpha_1} v_1$, and similarly the T -eigenspace of weight $(n-1)\omega_1$ in $V(\omega_1)^{\otimes n}$ is spanned by $v_1^{\otimes i-1} \otimes v_0 \otimes v_1^{\otimes n-i}$, where $1 \leq i \leq n$. Since the vectors

$$e_{\alpha_r}(v_1^{\otimes i-1} \otimes v_0 \otimes v_1^{\otimes n-i}) = v_1^{\otimes i-1} \otimes (e_{\alpha_r} v_0) \otimes v_1^{\otimes n-i}$$

are linearly independent, there exists no maximal vector of weight $(n-1)\omega_1$ in $V(\omega_1)^{\otimes n}$. \square

Dual results will be needed to describe the subsets Σ which give rise to a normal variety X_{Σ} , but before we need to introduce some further notation.

If Φ is an irreducible root system and Δ is a basis for Φ we will denote by η the highest root if Φ is simply laced or the highest short root if Φ is not simply laced. For the convenience of the reader we list the highest short root of every irreducible root system in Table 1.

Recall the condition (\star) defined in the introduction: a dominant weight λ satisfies (\star) if, for every non-simply laced connected component $\Delta' \subset \Delta$, if $\text{Supp}(\lambda) \cap \Delta'$ contains a long root then it contains also the short root which is adjacent to a long simple root.

Definition 10. If $\Delta' \subset \Delta$ is a non-simply laced connected component, order the simple roots in $\Delta' = \{\alpha_1, \dots, \alpha_r\}$ starting from the extreme of the Dynkin diagram of Δ' which contains a long root and denote α_q the first short root in Δ' . If λ is a dominant weight such that $\alpha_q \notin \text{Supp}(\lambda)$ and such that

$\text{Supp}(\lambda) \cap \Delta'$ contains a long root, denote α_p the last long root which occurs in $\text{Supp}(\lambda) \cap \Delta'$; for instance, if Δ' is not of type G_2 , then the numbering is as follows:

$$\begin{array}{ccccccc} & & & & \longrightarrow & & \\ \alpha_1 & \cdots & \alpha_p & \cdots & \alpha_q & \cdots & \alpha_r \end{array}$$

The *little brother* of λ with respect to Δ' is the dominant weight

$$\lambda_{\Delta'}^{\text{lb}} = \lambda - \sum_{i=p}^q \alpha_i = \begin{cases} \lambda - \omega_1 + \omega_2 & \text{if } G \text{ is of type } G_2 \\ \lambda + \omega_{p-1} - \omega_p + \omega_{q+1} & \text{otherwise} \end{cases}$$

where ω_i is the fundamental weight associated to α_i if $1 \leq i \leq r$, while $\omega_0 = \omega_{r+1} = 0$. The set of the little brothers of λ will be denoted by $\text{LB}(\lambda)$; notice that $\text{LB}(\lambda)$ is empty if and only if λ satisfies condition (\star) of Theorem A. For convenience, define $\overline{\text{LB}}(\lambda) = \text{LB}(\lambda) \cup \{\lambda\}$, while if Δ is connected and non-simply laced set $\lambda^{\text{lb}} = \lambda_{\Delta}^{\text{lb}}$.

Lemma 11. *Assume G to be simple and let $\lambda \in \Lambda^+ \setminus \{0\}$. Denote η the highest root of Φ if the latter is simply laced or the highest short root otherwise.*

(1) *If λ satisfies the condition (\star) then*

$$V(\lambda) \subset V(\eta) \otimes V(\lambda).$$

(2) *If λ does not satisfy the condition (\star) and if λ^{lb} is the little brother of λ then*

$$V(\lambda) \subset V(\eta) \otimes V(\lambda^{\text{lb}}).$$

Proof. If Δ is simply laced, then $V(\eta) \simeq \mathfrak{g}$ is the adjoint representation: in this case the claim follows straightforward by considering the map $\mathfrak{g} \otimes V(\lambda) \rightarrow V(\lambda)$ induced by the \mathfrak{g} -module structure on $V(\lambda)$, which is non-zero since λ is non-zero.

Suppose now that Δ is not simply laced. If λ satisfies condition (\star) , then by Lemma 8 it is enough to study the case $\lambda = \omega_\alpha$ where α is a short simple root:

Type B_r : $V(\omega_r) \subset V(\omega_1) \otimes V(\omega_r)$.

Type C_r : $V(\omega_i) \subset V(\omega_2) \otimes V(\omega_i)$, with $i < r$.

Type F_4 : $V(\omega_3) \subset V(\omega_4) \otimes V(\omega_3)$ and $V(\omega_4) \subset V(\omega_4) \otimes V(\omega_4)$.

Type G_2 : $V(\omega_1) \subset V(\omega_1) \otimes V(\omega_1)$.

If λ does not satisfy condition (\star) , by Lemma 8 we can assume that $\lambda = \omega_\alpha$ with α a long root:

Type B_r : $V(\omega_i) \subset V(\omega_1) \otimes V(\omega_{i-1})$, if $1 < i < r$, and $V(\omega_1) \subset V(\omega_1) \otimes V(0)$.

Type C_r : $V(\omega_r) \subset V(\omega_2) \otimes V(\omega_{r-2})$.

Type F_4 : $V(\omega_1) \subset V(\omega_4) \otimes V(\omega_4)$ and $V(\omega_2) \subset V(\omega_4) \otimes V(\omega_1 + \omega_4)$.

Type G_2 : $V(\omega_2) \subset V(\omega_1) \otimes V(\omega_1)$.

The above mentioned inclusion relations for tensor products are essentially known: let us treat the case of type C_r with $\lambda = \omega_i$ and $i < r$, the other cases are easier or can be checked directly.

Let v_0 be a highest weight vector of $V(\omega_2)$ and w_0 be a highest weight vector of $V(\omega_i)$. Let f be the following product (in the universal enveloping algebra $\mathfrak{U}(\mathfrak{u}^-)$)

$$f = f_{\alpha_i} \cdots f_{\alpha_1} \cdot f_{\alpha_{i+1}} \cdots f_{\alpha_{r-1}} \cdot f_{\alpha_r} \cdots f_{\alpha_2},$$

and consider all the factorizations $f = p \cdot q$ such that $p, q \in \mathfrak{U}(\mathfrak{u}^-)$. If $\beta_1, \dots, \beta_j \in \Delta$, set

$${}^r(f_{\beta_1} \cdots f_{\beta_j}) = (-1)^j 2^\delta f_{\beta_j} \cdots f_{\beta_1},$$

where δ equals 0 (resp. 1) if α_i occurs an even (resp. odd) number of times in $\{\beta_1, \dots, \beta_j\}$. Then it is easy to check that the vector

$$\sum_{p \cdot q = f} p \cdot v_0 \otimes {}^r q \cdot w_0$$

is a U -invariant vector in $V(\omega_2) \otimes V(\omega_i)$ of T -weight ω_i . □

If the Dynkin diagram of G is not simply laced we will need some further properties of tensor products.

If Δ is connected but not simply laced, we will denote by α_S the short simple root that is adjacent to a long simple root α_L ; moreover, we will denote the associated fundamental weights by ω_S and ω_L . Finally, define ζ as the sum of all simple roots and notice that $\omega_S + \zeta$ is dominant.

Lemma 12. *Let λ be a non-zero dominant weight.*

(1) *If G is of type F_4 or C_r ($r \geq 3$) and if $\text{Supp}(\lambda)$ contains a long root then*

$$V(\lambda + \omega_S) \subset V(\zeta + \omega_S) \otimes V(\lambda).$$

(2) *If G is of type G_2 and if λ does not satisfy (\star) then*

$$V(\lambda + \omega_1) \subset V(\omega_2) \otimes V(\lambda^{\text{lb}}).$$

(3) *If G is of type G_2 and if $\alpha_S \in \text{Supp}(\lambda)$ then*

$$V(\lambda + \omega_1) \subset V(\omega_2) \otimes V(\lambda).$$

Proof. By Lemma 8 it is enough to check the statements for $\lambda = \omega_\alpha$ with α a long root in the first two cases and $\alpha = \alpha_S$ in the last case.

Type C_r : by Lemma 8 it is enough to check that $V(\omega_{r-1}) \subset V(\omega_1) \otimes V(\omega_r)$.

Type F_4 : we have $\lambda = \omega_1$ or $\lambda = \omega_2$ and $\omega_S + \zeta = \omega_1 + \omega_4$.

Type G_2 : we have $\lambda = \omega_2$ and $\lambda^{\text{lb}} = \omega_1$ in point (2) and $\lambda = \omega_1$ in point (3). \square

2.2. Normality and non-normality of X_Σ . We are now able to state the main theorem.

Theorem 13. *Let Σ be a simple set of dominant weights and let λ be its maximal element. The variety X_Σ is normal if and only if $\Sigma \supset \text{LB}(\lambda)$.*

Theorem A stated in the introduction follows immediately by considering the case $\Sigma = \{\lambda\}$. The remaining part of this section will be devoted to the proof of Theorem 13. The general strategy will be based on Proposition 6 and will proceed by induction on the dominance order of weights. The ingredients of this induction will be the results proved in section 2.1 together with the description of the dominance order given by J. Stembridge in [St]: the dominance order between dominant weights is generated by pairs which differ by the highest short root for a subsystem of the root system.

If K is a subset of Δ , denote $\Phi_K \subset \Phi$ the associated root subsystem and, in case K is connected, denote by η_K the corresponding highest short root. Moreover, if $\beta = \sum_{\alpha \in \Delta} n_\alpha \alpha$, set $\beta|_K = \sum_{\alpha \in K} n_\alpha \alpha$. The result of [St] that we will use is the following.

Lemma 14 ([St, Lemma 2.5]). *Let λ, μ be two dominant weights with $\lambda > \mu$; set $I = \text{Supp}_\Delta(\lambda - \mu)$. Let Φ_K be an irreducible subsystem of Φ_I (where $K \subset I$).*

- (a) *If $\langle (\lambda - \mu)|_K, \alpha^\vee \rangle \geq 0$ for all $\alpha \in K \cap \text{Supp}(\mu)$, then $\mu + \eta_K \leq \lambda$.*
- (b) *If in addition $\langle \mu + \eta_K, \alpha^\vee \rangle \geq 0$ for all $\alpha \in I \setminus K$, then $\mu + \eta_K \in \Lambda^+$.*

The next two lemmas are the main steps of our induction.

Lemma 15. *Suppose that Φ is irreducible; let $\lambda, \mu \in \Lambda^+$ such that $\lambda > \mu$ and $\text{Supp}_\Delta(\lambda - \mu) = \Delta$. Assume that either Φ is simply laced, or there exists a short root $\alpha \in \text{Supp}(\lambda)$ such that $\langle \lambda - \mu, \alpha^\vee \rangle \geq 0$, or $\alpha_S \notin \text{Supp}(\mu)$. Then there exists a connected subset K of Δ such that*

- i) $\mu + \eta_K \leq \lambda$;
- ii) $\mu + \eta_K \in \Lambda^+$;
- iii) $K \cap \text{Supp}(\lambda) \neq \emptyset$.

Proof. Set $K_1 = \{\alpha \in \Delta : \langle \lambda - \mu, \alpha^\vee \rangle \geq 0\}$. Since $\lambda > \mu$ we have that $K_1 \cap \text{Supp}(\lambda)$ is non-empty. Notice also that $\text{Supp}(\mu) \supset \Delta \setminus K_1$. Define K as follows:

- a) If Φ is simply laced, let K be a connected component of K_1 which intersects $\text{Supp}(\lambda)$.
- b) If $\alpha \in \text{Supp}(\lambda)$ is a short root such that $\langle \lambda - \mu, \alpha^\vee \rangle \geq 0$ let K be the connected component of K_1 containing α .
- c) If Φ is not simply laced and there does not exist a short root α as in b), let K be a connected component of K_1 which intersects $\text{Supp}(\lambda)$.

Properties *i)* and *iii)* are then easily verified by Lemma 14(a) and by construction.

To prove *ii)* notice that, if Φ is not simply laced, by the construction of K it follows that if $\alpha_L \in K$ then $\alpha_S \in K$ as well: indeed, K is a connected component of K_1 and if there is no short root α as in b) then $\alpha_S \notin \text{Supp}(\mu)$ implies $\alpha_S \in K_1$. By the description of highest short roots in Table 1 we deduce that, if $\alpha \in K \setminus K^\circ$, then the respective coefficient in η_K is 1: hence $\langle \eta_K, \alpha^\vee \rangle = -1$ for all $\alpha \in \partial K$ and, since $\text{Supp}(\mu) \supset \Delta \setminus K_1 \supset \partial K$, we get $\mu + \eta_K \in \Lambda^+$. \square

In order to proceed with the induction, in the next lemma we will need to consider the condition (\star) also for a Levi subgroup of G . If $K \subset \Delta$ let L_K be the associated standard Levi subgroup; we say that $\lambda \in \Lambda^+$ satisfies condition (\star_K) if, for every non-simply laced connected component K' of K such that $\text{Supp}(\lambda) \cap K'$ contains a long root, $\text{Supp}(\lambda) \cap K'$ contains also the short root adjacent to a long root. Notice that if λ satisfies (\star) then it also satisfies (\star_K) for all $K \subset \Delta$.

Similarly we can also define the little brother of a dominant weight w.r.t. the Levi subgroup L_K : if K' is a connected component of K such that λ does not satisfy $(\star_{K'})$, define the little brother $\lambda_{K'}^{\text{lb}}$, w.r.t. K' as in Definition 10 and denote by $\text{LB}_K(\lambda)$ the set of little brothers of λ constructed in this way. Notice that if K' is a connected component of K such that λ does not satisfy $(\star_{K'})$ and if Δ' is the connected component of Δ containing K' , then λ does not satisfy $(\star_{\Delta'})$ as well and $\lambda_{K'}^{\text{lb}} = \lambda_{\Delta'}^{\text{lb}}$. In particular $\text{LB}_K(\lambda) \subset \text{LB}(\lambda)$.

Lemma 16. *Assume G to be simple and let λ, μ be two dominant weights such that $\lambda > \mu$ and $\text{Supp}_\Delta(\lambda - \mu) = \Delta$. Then there exist $\mu' \in \Lambda^+$ and $\lambda' \in \overline{\text{LB}}(\lambda)$ such that $\mu < \mu' \leq \lambda$ and*

$$V(\mu + \lambda) \subset V(\mu') \otimes V(\lambda').$$

Proof. Suppose first that either Φ is simply laced or $\alpha_S \notin \text{Supp}(\mu)$ or there exists a short root α in $\text{Supp}(\lambda)$ such that $\langle \lambda - \mu, \alpha^\vee \rangle \geq 0$. Take K as in Lemma 15 and set $\mu' = \mu + \eta_K$: then by Lemma 11 together with Lemma 8 we get

$$V_{L_K}(\mu + \lambda) \subset V_{L_K}(\mu') \otimes V_{L_K}(\lambda')$$

with $\lambda' \in \overline{\text{LB}}_K(\lambda)$. The claim follows by Lemma 7 together with the inclusion $\text{LB}_K(\lambda) \subset \text{LB}(\lambda)$.

Suppose now that Φ is not simply laced, that $\alpha_S \in \text{Supp}(\mu)$ and that there is no short root $\alpha \in \text{Supp}(\lambda)$ such that $\langle \lambda - \mu, \alpha^\vee \rangle \geq 0$. Since $\lambda > \mu$ there exists $\alpha \in \text{Supp}(\lambda)$ such that $\langle \lambda - \mu, \alpha^\vee \rangle > 0$: therefore, $\text{Supp}(\lambda)$ contains at least a long root. Set $\mu' = \mu + \zeta$; notice that $\mu' \leq \lambda$ and that μ' is dominant. The claim follows then by Lemma 11 and by Lemma 8 if Φ is of type \mathbf{B} , while if Φ is of type \mathbf{C} , \mathbf{F}_4 or \mathbf{G}_2 it follows by Lemma 12 and by Lemma 8. \square

Proof of Theorem 13. We prove first that the condition is necessary. Assume that there exists a little brother $\mu = \lambda_{\Delta'}^{\text{lb}}$ of λ which is not in Σ . We prove that for every positive n and for every choice of weights $\lambda_1, \dots, \lambda_n \in \Sigma$ the module $V(\mu + (n-1)\lambda)$ is not contained in $V(\lambda_1) \otimes \dots \otimes V(\lambda_n)$.

We proceed by contradiction. Assume there exist weights $\lambda_1, \dots, \lambda_n$ as above and notice that any of them satisfies $\mu \leq \lambda_i \leq \lambda$: indeed, $\lambda - \mu = n\lambda - (\mu + (n-1)\lambda) \geq n\lambda - \sum \lambda_i \geq \lambda - \lambda_i$ for every i . Therefore $\text{Supp}_\Delta(\sum \lambda_i - (\mu + (n-1)\lambda)) \subset \text{Supp}_\Delta(\lambda - \mu)$. By Definition 10 together with Lemma 7, it is enough to analyse the case G of type \mathbf{B}_r and $\text{Supp}(\lambda) = \{\alpha_1\}$ or G of type \mathbf{G}_2 and $\text{Supp}(\lambda) = \{\alpha_2\}$. We analyse these two cases separately.

Type \mathbf{B}_r : we have $\lambda = a\omega_1$, $\mu = (a-1)\omega_1$ and $\mu + (n-1)\lambda = (na-1)\omega_1$. If $a = 1$ we notice that there are no dominant weights between λ and μ . So the only possibility is $\lambda_i = \lambda = \omega_1$ for all

i and this is in contradiction with Lemma 9. If $a > 1$, notice that there is only one dominant weight between λ and μ , namely $\nu = \lambda - \alpha_1 = (a-2)\omega_1 + \omega_2$; hence for all i it must be $\lambda_i = \lambda$ or $\lambda_i = \nu$. Since $\sum \lambda_i \geq \mu + (n-1)\lambda$, at most one λ_i can be equal to ν ; therefore $V(\mu + (n-1)\lambda) \subset V(\lambda)^{\otimes n}$ or $V(\mu + (n-1)\lambda) \subset V(\nu) \otimes V(\lambda)^{\otimes(n-1)}$. In the first case we obtain

$$V((na-1)\omega_1) = V(\mu + (n-1)\lambda) \subset V(\lambda)^{\otimes n} \subset V(\omega_1)^{\otimes na},$$

against Lemma 9. In the second case we notice that $V(\omega_2) = \Lambda^2 V(\omega_1) \subset V(\omega_1)^{\otimes 2}$, hence $V(\nu) \subset V((a-2)\omega_1) \otimes V(\omega_2) \subset V(\omega_1)^{\otimes a}$ and we can conclude as in the first case.

Type G₂: we have $\lambda = a\omega_2$, $\mu = \omega_1 + (a-1)\omega_2$ and we proceed as in the previous case.

We now prove that the condition is sufficient, showing that for every dominant weight $\mu \leq \lambda$ there exist $n > 0$ and weights $\lambda_1, \dots, \lambda_n \in \overline{\text{LB}}(\lambda)$ such that $V(\mu + (n-1)\lambda) \subset V(\lambda_1) \otimes \dots \otimes V(\lambda_n)$. To do this, we proceed by decreasing induction with respect to the dominance order.

If $\mu = \lambda$ then the claim is clear, so we assume $\mu < \lambda$. Let $\lambda - \mu = \beta_1 + \dots + \beta_m$ where $\text{Supp}_\Delta(\beta_i)$ are the connected components of $\text{Supp}_\Delta(\lambda - \mu)$. Set $K = \text{Supp}_\Delta(\beta_1)$ and $\beta' = \beta_2 + \dots + \beta_m$. Notice that $\mu + \beta_1$ is dominant: indeed if $\alpha \notin \overline{K}$ then $\langle \mu + \beta_1, \alpha^\vee \rangle = \langle \mu, \alpha^\vee \rangle \geq 0$, while if $\alpha \in \overline{K}$ then $\langle \mu + \beta_1, \alpha^\vee \rangle = \langle \lambda - \beta', \alpha^\vee \rangle \geq \langle \lambda, \alpha^\vee \rangle \geq 0$. Notice moreover that, if $\nu \in \overline{\text{LB}}_K(\mu + \beta_1)$, then $\nu + \beta' \in \overline{\text{LB}}(\lambda)$. By Lemma 16 applied to the semisimple part of the Levi $L = L_K$ associated to K , there exists a weight μ' which is dominant with respect to K such that $\mu < \mu' \leq \mu + \beta_1$ and there exists $\nu \in \overline{\text{LB}}_K(\mu + \beta_1)$ which satisfy

$$V_L(\mu + \beta_1 + \mu) \subset V_L(\mu') \otimes V_L(\nu).$$

By tensoring with $V_L(\beta')$, which is a one dimensional representation, we get $V_L(\mu + \lambda) \subset V_L(\mu') \otimes V_L(\lambda')$ with $\lambda' = \nu + \beta' \in \overline{\text{LB}}(\lambda)$. Since $\langle \mu', \alpha^\vee \rangle \geq \langle \mu + \beta_1, \alpha^\vee \rangle$ for every $\alpha \notin K$, μ' is a dominant weight; by Lemma 7 we get then $V(\mu + \lambda) \subset V(\mu') \otimes V(\lambda')$ and we may apply the induction on $\mu' \leq \lambda$. Therefore there exist weights $\lambda_1, \dots, \lambda_n \in \overline{\text{LB}}(\lambda)$ such that $V(\mu' + (n-1)\lambda) \subset V(\lambda_1) \otimes \dots \otimes V(\lambda_n)$. Finally by Lemma 8 we conclude

$$V(\mu + n\lambda) \subset V(\mu' + (n-1)\lambda) \otimes V(\lambda') \subset V(\lambda_1) \otimes \dots \otimes V(\lambda_n) \otimes V(\lambda').$$

□

3. SMOOTHNESS

In this section we will study the variety \tilde{X}_λ ; in particular we will give necessary and sufficient conditions on $\text{Supp}(\lambda)$ for its \mathbb{Q} -factoriality and for its smoothness.

Thanks to Lemma 1, we may assume that G is a simple group. Indeed suppose $\Delta = \cup_{i=1}^n \Delta_i$ is the decomposition in connected components and write $\lambda = \lambda_1 + \dots + \lambda_n$ with $\text{Supp}(\lambda_i) \subset \Delta_i$: correspondingly we get a decomposition $X_\lambda = X_{\lambda_1} \times \dots \times X_{\lambda_n}$, and every X_{λ_i} is an embedding of the corresponding simple factor of G_{ad} if $\lambda_i \neq 0$ or a point if $\lambda_i = 0$. From now on, we will therefore assume that Φ is an irreducible root system.

By the Bruhat decomposition, the group G_{ad} has an open $B \times B^-$ -orbit; therefore it is a spherical $G \times G$ -homogeneous space. Following the general theory of spherical embeddings (see [Kn]), its simple normal embeddings are classified by combinatorial data called the *colored cones*. Here we will skip an overview of such theory, and we will simply recall the definition of the colored cone in the particular case of a simple normal embedding of G_{ad} .

Recall that a normal variety X is said *\mathbb{Q} -factorial* if, given any Weil divisor D of X , there exists an integer $n \neq 0$ such that nD is a Cartier divisor. In subsection 3.1, we will explicitly describe the colored cone of \tilde{X}_λ ; then in subsection 3.2 we will study \mathbb{Q} -factoriality of \tilde{X}_λ following [Br]. Finally, in subsection 3.3, we will use Theorem 13 together with the description of the colored cone of \tilde{X}_λ to make

more explicit the criterion of smoothness given in [Ti] in the case of a linear projective compactification of a reductive group.

3.1. The colored cone of \tilde{X}_λ . Let X be a simple normal compactification of G_{ad} , call Y its unique closed orbit. Set $\mathcal{D}(G_{\text{ad}})$ the set of $B \times B^-$ -stable prime divisors of G_{ad} and $\mathcal{D}(X) \subset \mathcal{D}(G_{\text{ad}})$ the set of divisors whose closure in X contains Y . Let $\mathcal{N}(X)$ be the set of $G \times G$ -stable prime divisors of X , so that the set of $B \times B^-$ -stable prime divisors of X is identified with $\mathcal{D}(G_{\text{ad}}) \cup \mathcal{N}(X)$.

Let $T_{\text{ad}} \subset G_{\text{ad}}$ be the image of T ; then the character group $\mathcal{X}(T_{\text{ad}})$ coincides with the root lattice $\mathbb{Z}\Delta$, while the cocharacter group $\mathcal{X}^\vee(T_{\text{ad}})$ coincides with the coweight lattice Λ^\vee . If V is a simple $G \times G$ -module denote by $V^{(B \times B^-)}$ the subset of $B \times B^-$ -eigenvectors. Notice that $\mathbb{k}(G_{\text{ad}})^{(B \times B^-)}/\mathbb{k}^* \simeq \mathbb{Z}\Delta$ and define a natural map $\rho : \mathcal{D}(G_{\text{ad}}) \cup \mathcal{N}(X) \rightarrow \Lambda^\vee$ by associating to a $B \times B^-$ -stable prime divisor of X the cocharacter associated to the rational discrete valuation induced by D . If $D \in \mathcal{N}(X)$, then $\rho(D)$ is the opposite of a fundamental coweight, while if $D \in \mathcal{D}(G_{\text{ad}})$, then $\rho(D)$ is a simple coroot; moreover, ρ is injective and $\rho(\mathcal{D}(G_{\text{ad}})) = \Delta^\vee$ (see [Ti, § 7]).

Let $\mathcal{C}(X)$ be the convex cone in $\Lambda_{\mathbb{Q}}^\vee$ generated by $\rho(\mathcal{D}(X) \cup \mathcal{N}(X))$; by the general theory of spherical embeddings we have that $\mathcal{C}(X)$ is generated by $\rho(\mathcal{D}(X))$ together with the negative Weyl chamber of Φ . The *colored cone* of X is then the couple $(\mathcal{C}(X), \mathcal{D}(X))$: up to equivariant isomorphisms, it uniquely determines X as a $G \times G$ -compactification of G_{ad} .

In the case of the compactification \tilde{X}_λ , then $\rho(\mathcal{D}(X)) = \Delta^\vee \setminus \text{Supp}(\lambda)^\vee$ (see [Ti, Theorem 7]).

3.2. \mathbb{Q} -factoriality. In order to give a necessary and sufficient condition for the \mathbb{Q} -factoriality of \tilde{X}_λ we need to determine the set of extremal rays of the associated cone $\mathcal{C}(\tilde{X}_\lambda)$.

Lemma 17. *If $\alpha \in \Delta \setminus \text{Supp}(\lambda)$, then α^\vee generates an extremal ray of $\mathcal{C}(\tilde{X}_\lambda)$.*

Proof. If a simple coroot $\alpha^\vee \in \mathcal{C}(\tilde{X}_\lambda)$ does not generate an extremal ray, then we can write

$$\alpha^\vee = \sum_{\beta \in \Delta \setminus \{\alpha\}} a_\beta \beta^\vee - \sum_{\beta \in \Delta} b_\beta \omega_\beta^\vee,$$

with $a_\beta, b_\beta \geq 0$ for every β : this yields a contradiction since then it would be $\langle \alpha, \alpha^\vee \rangle \leq 0$. \square

Recall that a convex cone is said to be *simplicial* if it is generated by linearly independent vectors; the following proposition is a particular case of a characterization of \mathbb{Q} -factoriality that M. Brion gave in [Br] in the general case of a spherical variety. We recall it in the case of our interest.

Proposition 18 (see [Br, Proposition 4.2]). *The variety \tilde{X}_λ is \mathbb{Q} -factorial if and only if $\mathcal{C}(\tilde{X}_\lambda)$ is simplicial.*

Therefore, since $\mathcal{C}(\tilde{X}_\lambda)$ has maximal dimension, \tilde{X}_λ is \mathbb{Q} -factorial if and only if the number of extremal rays of the associated cone equals the rank of G . To describe such rays we need to introduce some more notation; the description will be slightly more complicated if Φ is of type D or E.

Denote Δ^e the set of extremal roots of Δ and set $\Delta \setminus \text{Supp}(\lambda) = \bigcup_{i=1}^n I_i$ the decomposition in connected components. Denote

$$I^e = \bigcup_{\substack{I_i \neq I_{\text{de}} \\ I_i \cap \Delta^e \neq \emptyset}} I_i,$$

where I_{de} is defined as follows. If Δ is of type D or E, denote γ_{de} the unique simple root which is adjacent to other three simple roots and, if it exists, denote $I_{\text{de}} \subset \Delta \setminus \text{Supp}(\lambda)$ the unique connected component such that $\gamma_{\text{de}} \in I_{\text{de}}$ and $|I_{\text{de}} \cap \Delta^e| = 1$, otherwise define I_{de} to be the empty set. Denote $I_{\text{de}}^* \subset I_{\text{de}}$ the minimal connected subset such that $\gamma_{\text{de}} \in I_{\text{de}}^*$ and $I_{\text{de}}^* \cap \Delta^e \neq \emptyset$, or define it to be the empty set otherwise. Finally define

$$J(\lambda) = (\Delta \setminus (\overline{I^e} \cup I_{\text{de}}^*)) \cup (\Delta^e \setminus \text{Supp}(\lambda)).$$

Lemma 19. *The extremal rays of $\mathcal{C}(\tilde{X}_\lambda)$ are generated by the simple coroots α^\vee with $\alpha \in \Delta \setminus \text{Supp}(\lambda)$ and by the opposite of fundamental coweights $-\omega_\alpha^\vee$ with $\alpha \in J(\lambda)$.*

Proof. Recall that $\mathcal{C}(\tilde{X}_\lambda)$ is generated by the simple coroots α^\vee with $\alpha \in \Delta \setminus \text{Supp}(\lambda)$ together with the fundamental coweights $-\omega_\alpha^\vee$ with $\alpha \in \Delta$ and that every coroot α^\vee with $\alpha \in \Delta \setminus \text{Supp}(\lambda)$ generates an extremal ray of $\mathcal{C}(\tilde{X}_\lambda)$.

A coweight $-\omega_\alpha^\vee$ does not generate an extremal ray if and only if it can be written as follows

$$-\omega_\alpha^\vee = \sum_{\beta \in K} a_\beta \beta^\vee - \sum_{\beta \in H} b_\beta \omega_\beta^\vee$$

with $a_\beta > 0$ for every $\beta \in K$ and with $b_\beta > 0$ for every $\beta \in H$, for suitable non-empty subsets $K \subset \Delta \setminus \text{Supp}(\lambda)$ and $H \subset \Delta \setminus \{\alpha\}$. Since the right member of the equality is negative against every simple root in ∂K , we get $\partial K = \{\alpha\}$.

Notice that K is connected. Indeed if $K' \subset K$ is a connected component then $\partial K' = \{\alpha\}$ and $\sum_{\beta \in K'} a_\beta \langle \alpha, \beta^\vee \rangle < 0$: therefore if K contains two connected components it must be

$$\sum_{\beta \in K} a_\beta \langle \alpha, \beta^\vee \rangle \leq -2.$$

On the other hand $\langle \alpha, \omega_\beta^\vee \rangle = 0$ for every $\beta \in H$, therefore if K is not connected it follows

$$-1 = -\langle \alpha, \omega_\alpha^\vee \rangle = \sum_{\beta \in K} a_\beta \langle \alpha, \beta^\vee \rangle \leq -2.$$

Since ∂K is one single root, K contains an extreme of Δ , thus we get $K \subset I^e \cup I_{\text{de}}$. Suppose that $\gamma_{\text{de}} \in K \subset I_{\text{de}}$: then we get a contradiction since it would be $|\partial K| = 2$. Therefore we get $K \subset I^e \cup (I_{\text{de}}^* \setminus \{\gamma_{\text{de}}\})$ and $\alpha \in \overline{I^e} \cup I_{\text{de}}^*$. Such a subset K cannot exist if $\alpha \in \Delta^e \setminus \text{Supp}(\lambda)$, otherwise it would be $K = \Delta \setminus \{\alpha\}$ which intersects $\text{Supp}(\lambda)$. We get then that every $-\omega_\alpha^\vee$ with $\alpha \in J(\lambda)$ generates an extremal ray of $\mathcal{C}(\tilde{X}_\lambda)$.

Suppose conversely that $\alpha \notin J(\lambda)$. Then we can construct a connected subset $K \subset I^e \cup (I_{\text{de}}^* \setminus \{\gamma_{\text{de}}\})$ such that $\partial K = \{\alpha\}$. If $\gamma \in K \cap \Delta^e$, consider the fundamental coweight $(\omega_\gamma^K)^\vee$ associated to γ in the irreducible root subsystem associated to K : then we get

$$(\omega_\gamma^K)^\vee = \sum_{\beta \in K} a_\beta \beta^\vee = \omega_\gamma^\vee - m\omega_\alpha^\vee,$$

where $a_\beta > 0$ are rational coefficients and where $m > 0$ is an integer. Therefore $-\omega_\alpha^\vee$ does not generate an extremal ray of $\mathcal{C}(\tilde{X}_\lambda)$. \square

Proposition 20. *The variety \tilde{X}_λ is \mathbb{Q} -factorial if and only if the following conditions are fulfilled:*

- i) $\text{Supp}(\lambda)$ is connected;
- ii) If $\text{Supp}(\lambda)$ contains a unique element, then this element is an extreme of Δ ;
- iii) If Δ is of type D or E, then $\text{Supp}(\lambda)$ contains γ_{de} and at least two simple roots adjacent to γ_{de} .

Proof. By Proposition 18 together with Lemma 19 \tilde{X}_λ is \mathbb{Q} -factorial if and only if $|\text{Supp}(\lambda)| = |J(\lambda)|$.

Suppose that \tilde{X}_λ is \mathbb{Q} -factorial. Consider the dominant weight $\lambda' = \sum_{\alpha \notin I^e \cup I_{\text{de}}^*} \omega_\alpha$: then $J(\lambda') = J(\lambda)$ and

$$|\Delta| = |J(\lambda)| + |\Delta \setminus \text{Supp}(\lambda)| \geq |J(\lambda')| + |\Delta \setminus \text{Supp}(\lambda')| \geq |\Delta|,$$

which implies $\text{Supp}(\lambda) = \text{Supp}(\lambda')$. This shows $\Delta \setminus \text{Supp}(\lambda) = I^e \cup I_{\text{de}}^*$, and we get the following decomposition of $J(\lambda)$:

$$J(\lambda) \cap \text{Supp}(\lambda) = \Delta \setminus (\overline{I^e} \cup I_{\text{de}}^*), \quad J(\lambda) \setminus \text{Supp}(\lambda) = \Delta^e \setminus \text{Supp}(\lambda).$$

If $I_{\text{de}} \neq \emptyset$, set $I_{\text{de}} \cap \Delta^e = \{\alpha_{\text{de}}\}$. Define a surjective map $F : J(\lambda) \setminus \{\alpha_{\text{de}}\} \rightarrow \text{Supp}(\lambda)$ as follows: F is the identity on $J(\lambda) \cap \text{Supp}(\lambda)$, while if $\alpha \in J(\lambda) \setminus \text{Supp}(\lambda)$ consider the connected component $K \subset \Delta \setminus \text{Supp}(\lambda)$ containing α and define $F(\alpha)$ by the relation $\partial K = \{F(\alpha)\}$: since $\alpha \neq \alpha_{\text{de}}$, it must

be $|\partial K| = 1$. Therefore F is well defined and it is surjective since $\text{Supp}(\lambda) \setminus J(\lambda) = \partial I^e$. Therefore $\Delta \setminus \text{Supp}(\lambda) = I^e$ and we get i). Being surjective, F has to be injective as well; this easily implies both ii) and iii).

Suppose conversely that $\text{Supp}(\lambda)$ is connected, or equivalently that $\Delta \setminus \text{Supp}(\lambda) = I^e$: then ii) and iii) imply $|\Delta^e \setminus \text{Supp}(\lambda)| = |\partial I^e|$. This shows that \tilde{X}_λ is \mathbb{Q} -factorial, since then $|J(\lambda)| + |\Delta \setminus \text{Supp}(\lambda)| = |\Delta|$. \square

Corollary 21. *If \tilde{X}_λ is \mathbb{Q} -factorial, the extremal rays of $\mathcal{C}(\tilde{X}_\lambda)$ are generated by:*

- i) the coroots α^\vee with $\alpha \in \Delta \setminus \text{Supp}(\lambda)$,
- ii) the coweights $-\omega_\alpha^\vee$ with $\alpha \in \text{Supp}(\lambda)^\circ \cup (\Delta^e \setminus \text{Supp}(\lambda))$.

3.3. Smoothness. Suppose that $\Sigma = \{\lambda, \lambda_1, \dots, \lambda_s\}$ is a simple set of dominant weights, where λ is the maximal one. In this section we will prove the following generalization of Theorem B.

Theorem 22. *The variety X_Σ is smooth if and only if X_λ is normal, \mathbb{Q} -factorial and every connected component of $\Delta \setminus \text{Supp}(\lambda)$ has type A.*

Corollary 23. *X_Σ is smooth if and only if X_λ is smooth.*

To prove Theorem 22, we will make use of a characterization of smoothness for arbitrary group compactifications given by D. Timashev in [Ti]. For convenience, we will use a generalization of it which can be found in [Ru] in the more general context of symmetric spaces. We recall it in the case of a simple group compactification.

Theorem 24 (see [Ru, Theorem 2.2], [Ti, Theorem 9]). *The variety \tilde{X}_λ is smooth if and only if the following conditions are fulfilled:*

- i) *All connected components of $\Delta \setminus \text{Supp}(\lambda)$ are of type A and there are no more than $|\text{Supp}(\lambda)|$ of them.*
- ii) *The cone $\mathcal{C}(\tilde{X}_\lambda)$ is simplicial and it is generated by a basis of the coweight lattice Λ^\vee .*
- iii) *One can enumerate the simple roots in order of their positions at Dynkin diagrams of connected components $I_k = \{\alpha_1^k, \dots, \alpha_{n_k}^k\}$ of $\Delta \setminus \text{Supp}(\lambda)$, $k = 1, \dots, n$, and partition the basis of the free semigroup $\mathcal{C}(\tilde{X}_\lambda)^\vee \cap \mathbb{Z}\Delta$ into subsets $\{\pi_1^k, \dots, \pi_{n_k+1}^k\}$, $k = 1, \dots, p$, $p \geq n$, in such a way that $\langle \pi_j^k, (\alpha_i^h)^\vee \rangle = \delta_{i,j} \delta_{h,k}$ and $\pi_j^k - \frac{j}{n_k+1} \pi_{n_k+1}^k$ is the j -th fundamental weight of the root system generated by $\{\alpha_1^k, \dots, \alpha_{n_k}^k\}$ for all j, k .*

Proof of Theorem 22. First, we prove that the conditions are necessary; since we only have to prove that X_λ is normal, we may assume that Δ is non-simply laced. By Theorem 24 i), $\text{Supp}(\lambda)$ contains at least one of the two simple roots α_S, α_L ; suppose that $\text{Supp}(\lambda)$ contains α_L but not α_S . Denote $K = \{\alpha_1, \dots, \alpha_l\} \subset \Delta \setminus \text{Supp}(\lambda)$ the connected component which contains α_S and number its simple roots starting from α_S : therefore $\alpha_1 = \alpha_S$ and $\alpha_l \in \Delta^e$, moreover \overline{K} is either of type C_{l+1} or of type G_2 . Set $\omega^\vee = (l+1)(\omega_l^K)^\vee$, where $(\omega_l^K)^\vee$ is the fundamental coweight associated to α_l in the root subsystem Φ_K associated to K ; then

$$\omega^\vee = \sum_{i=1}^l i \alpha_i^\vee = (l+1) \omega_{\alpha_l}^\vee - m \omega_{\alpha_L}^\vee.$$

where $m = 2$ if \overline{K} is of type C_{l+1} (with $l \geq 1$) and $m = 3$ if \overline{K} is of type G_2 .

If \overline{K} is not of type B_2 , then Δ is either of type C_r (with $r > 2$) or of type F_4 or of type G_2 and every simple coroot $\beta^\vee \in \Delta^\vee$ is a primitive element in Λ^\vee (i.e. there does not exist $\pi^\vee \in \Lambda^\vee$ which satisfies $t\pi^\vee = \beta^\vee$ with $t > 1$): therefore by Lemma 19 together with Theorem 24 ii) $\{\alpha_1^\vee, \dots, \alpha_l^\vee, \omega_{\alpha_l}^\vee\}$ is part of a basis of Λ^\vee and we get a contradiction since then the equality above would imply $\omega_{\alpha_L}^\vee \notin \Lambda^\vee$. Otherwise \overline{K} is of type B_2 , thus Δ is of type B_r and $\frac{1}{2}\alpha_S^\vee \in \Lambda^\vee$: then we get a contradiction since by Theorem 24 iii) there exists $\pi \in \mathcal{C}(\tilde{X}_\lambda)^\vee \cap \mathbb{Z}\Delta$ such that $\langle \pi, \alpha_S^\vee \rangle = 1$.

Let's prove now that conditions of Theorem 24 are verified if X_λ is normal, \mathbb{Q} -factorial and $\Delta \setminus \text{Supp}(\lambda)$ has type A. Set $N = \mathcal{C}(\tilde{X}_\lambda) \cap \Lambda^\vee$ the monoid generated by the primitive elements of the extremal rays of $\mathcal{C}(\tilde{X}_\lambda)$.

To prove condition i), it is enough to notice as in Proposition 20 that, since $\text{Supp}(\lambda)$ is connected, we have $\Delta \setminus \text{Supp}(\lambda) = I^e$ and the number of its connected components equals $|\Delta^e \setminus \text{Supp}(\lambda)| \leq |J(\lambda)| = |\text{Supp}(\lambda)|$.

To prove condition ii), let's show that, if $\beta \in \Delta \setminus J(\lambda) = \overline{I^e} \setminus \Delta^e$, then $-\omega_\beta^\vee \in N$. Denote $I = \{\alpha_1, \dots, \alpha_l\} \subset \Delta \setminus \text{Supp}(\lambda)$ the connected component which contains β in its closure and number its simple roots starting from the extreme of I which is not an extreme of Δ ; therefore $\alpha_l \in \Delta^e$. Let j be such that $\beta = \alpha_j$ or set $j = 0$ if $\beta \in \text{Supp}(\lambda)$. Set $K = \{\alpha_{j+1}, \dots, \alpha_l\}$ and set $\omega^\vee = (l - j + 1)(\omega_l^K)^\vee$, where $(\omega_l^K)^\vee$ is the fundamental weight associated to α_l in the root subsystem Φ_K associated to K ; then

$$\omega^\vee = \sum_{i=1}^{l-j} i\alpha_{j+i}^\vee = (l - j + 1)\omega_{\alpha_l}^\vee + \langle \beta, \alpha_{j+1}^\vee \rangle \omega_\beta^\vee.$$

Since X_λ is normal, by Theorem A we get $\langle \beta, \alpha_{j+1}^\vee \rangle = -1$; therefore by Corollary 21 $-\omega_\beta^\vee \in N$.

Finally let's show that condition iii) holds. Suppose that $K = \{\alpha_1, \dots, \alpha_l\} \subset \Delta \setminus \text{Supp}(\lambda)$ is a connected component, where the simple roots in K are numbered starting from the extreme of K which is not an extreme of Δ , and define

$$\pi_i^K = \begin{cases} (\alpha_i^\vee)^* & \text{if } i \leq l \\ (-\omega_{\alpha_l}^\vee)^* & \text{if } i = l + 1 \end{cases}$$

where, if $\{v_1, \dots, v_r\}$ is a basis of Λ^\vee , $\{v_1^*, \dots, v_r^*\}$ denotes the dual basis of Λ . Therefore, if ω_j^K is the j -th fundamental weight of Φ_K , we have $\omega_j^K = \pi_j^K - \frac{j}{l+1}\pi_{l+1}^K$. \square

4. REMARKS AND GENERALIZATIONS

In this section we will consider the more general situation of compactifications of symmetric varieties.

Let G be as before and $\sigma : G \rightarrow G$ an involution of G . We denote by H° the subgroup of points fixed by σ and by H its normalizer. The notation is not completely coherent with those of previous sections: G plays now the role that $G \times G$ played before, while H° has now the role that the diagonal of $G \times G$ had before.

Let Ω^+ be the set of dominant weights λ such that $V(\lambda)$ has a non-zero vector fixed by H° and Ω the sublattice of Λ generated by Ω^+ . The monoid Ω^+ (resp. the lattice Ω) is in a natural way the set of dominant weights (resp. the set of weights) of a (possibly non-reduced) root system $\tilde{\Phi}$, which is called the *restricted root system*. For $\lambda \in \Omega^+$ we can consider the (unique) point $x_\lambda \in \mathbb{P}(V(\lambda))$ fixed by H and define X_λ as the closure of the G -orbit of x_λ in $\mathbb{P}(V(\lambda))$.

Proposition 2 generalizes to this more general situation without any further comment.

4.1. Normality of X_λ and the closure of a maximal torus orbit. Let $T \subset G$ be a maximal torus such that the dimension of TH is maximal and let $Z_\lambda = \overline{Tx_\lambda} \subset X_\lambda$. In [Ru], it is proved that when X_λ is normal then Z_λ also is normal. The converse of this result does not hold in general. Indeed Z_λ is always normal in the case of the $G \times G$ -compactification of G_{ad} .

4.2. Generalization to symmetric varieties: normality. The wonderful compactification has been defined in the more general situation of symmetric varieties and the description of the normalization of X_λ generalizes thanks to the results contained in [CM] and [CDM] (which generalize [Ka] and [D]). In particular, Lemma 4 holds here in general. However, in the case of symmetric varieties we do not have a clear description of the multiplication of sections as in Lemma 5. In particular, we have no analogue of Proposition 6.

One may wonder whether the normality of X_λ is equivalent to the analogous combinatorial condition on the weight λ , that is, λ satisfies condition (\star) w.r.t. the root system $\tilde{\Phi}$; here is a counterexample.

Let G be of type B_2 and let σ be the involution of type B I: thus $G/H \simeq \mathrm{SO}(5)/\mathrm{S}(\mathrm{O}(3) \times \mathrm{O}(2))$ and $\tilde{\Delta} = 2\Delta$. Consider $\lambda = 2\omega_1 \in \Omega^+$; then X_λ is a normal embedding of G/H .

Denote by \leq_σ the dominance order w.r.t. the root system $\tilde{\Phi}$ and suppose that X_λ is normal. Then λ satisfies

$$\text{for all } \mu \in \Omega^+ \text{ such that } \mu \leq_\sigma \lambda \text{ there exists } n \in \mathbb{N} \text{ such that } V(\mu + (n-1)\lambda) \subset S^n(V(\lambda)).$$

If one assumes that the multiplication map is as generic as possible, then also the converse is true.

4.3. Generalization to symmetric varieties: smoothness. In the setting of normal compactifications of symmetric varieties G/H° , fix a maximal torus T such that TH° has maximal dimension and a Borel subgroup $B \supset T$ such that $BH^\circ \subset G$ is dense. If X is a simple normal compactification of G/H , denote $\mathcal{D}(X)$ the set of B -stable and not G -stable prime divisors of X which contain the closed orbit. Denote $\rho : \mathcal{D}(X) \rightarrow \Omega^\vee$ the map defined by the evaluation of functions; by [Vu, Proposition 1] $\rho(\mathcal{D}(X))$ is a basis of the restricted coroot system $\tilde{\Phi}^\vee$. Since the map ρ is not always injective, following the criterion of \mathbb{Q} -factoriality in [Br] in order to generalize Proposition 20 we only need to assume that ρ is injective on $\mathcal{D}(X)$, and the proof is the same. Such proposition is true also for compactifications of G/H° , and not only of G/H , since \mathbb{Q} -factoriality concerns no integrality questions.

Theorem 22 also can be generalized to this setting with the same proof, but we do not have anymore the equivalence between property (\star) and the normality of X_λ . Thus the theorem has to be reformulated as follows (recall that a simple normal spherical variety is always quasi-projective).

Theorem 25. *A simple normal compactification X of G/H is smooth if and only if it is \mathbb{Q} -factorial, $\Delta \setminus \rho(\mathcal{D}(X))$ satisfies (\star) and every connected component of $\rho(\mathcal{D}(X))$ has type A.*

REFERENCES

- [Bo] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitres IV, V, VI*, Actualités Scientifiques et Industrielles **1337**, Hermann Paris 1968.
- [Br] M. Brion, *Variétés sphériques et théorie de Mori*, Duke Math. J. **72** (1993) no. 2, 369–404.
- [CDM] R. Chirivì, C. De Concini and A. Maffei, *On normality of cones over symmetric varieties*, Tohoku Math. J. (2) **58** (2006) no. 4, 599–616.
- [CM] R. Chirivì and A. Maffei, *Projective normality of complete symmetric varieties*, Duke Math. J. **122** (2004), 93–123.
- [D] C. De Concini, *Normality and non normality of certain semigroups and orbit closures*, Algebraic transformation groups and algebraic varieties, Encyclopaedia Math. Sci. **132**, Springer Berlin 2004, 15–35.
- [DP] C. De Concini and C. Procesi, *Complete symmetric varieties*, Invariant Theory, Lecture Notes in Math. **996**, Springer, Berlin, 1983, 1–44.
- [Ka] S.S. Kannan, *Projective normality of the wonderful compactification of semisimple adjoint groups*, Math. Z. **239** (2002) 673–682.
- [Kn] F. Knop, *The Luna-Vust theory of spherical embeddings*, Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989), 225–249, Manoj Prakashan, Madras, 1991.
- [KKLV] F. Knop, H. Kraft, D. Luna and T. Vust, *Local properties of algebraic group actions*, Algebraische Transformationsgruppen und Invariantentheorie, DMV Sem. **13**, Birkhäuser, Basel, 1989, 63–75.
- [Ru] A. Ruzzi, *Smooth projective symmetric varieties with Picard number equal to one*. To appear in Internat. J. Math.
- [St] J.R. Stembridge, *The partial order of dominant weights*, Adv. Math. **136** (1998) no. 2, 340–364.
- [Ti] D.A. Timashev, *Equivariant compactifications of reductive groups*, Sb. Math. **194** (2003) no. 3–4, 589–616.
- [Vu] Th. Vust, *Plongements d’espaces symétriques algébriques: une classification*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **17** (1990), no. 2, 165–195.

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